On Stueckelberg-Horwitz-Piron Gravitation And Kaluza-Klein Theory*

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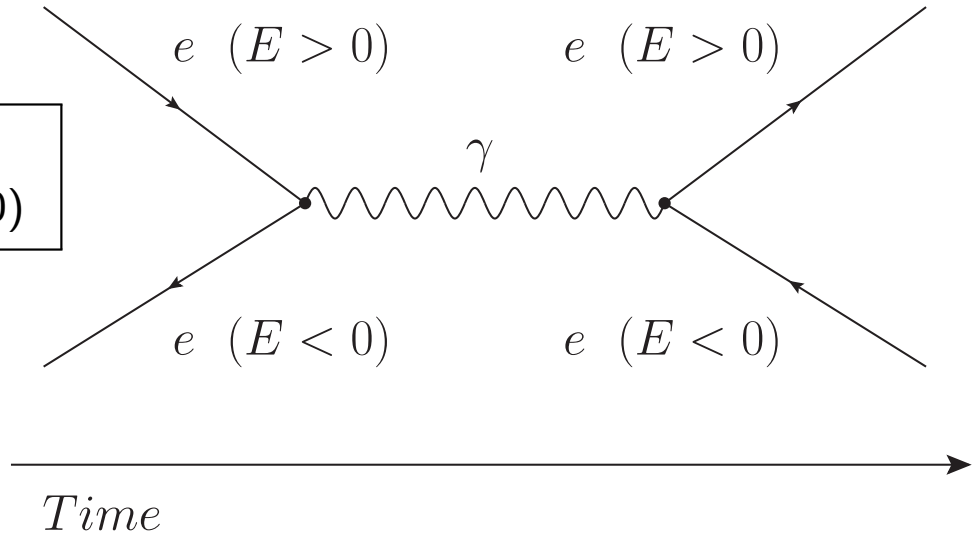
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Pair processes in Standard Model

Bhabha scattering (1936)
Feynman diagram (1948, 1950)



Annihilation

Electron with $E > 0$ emits photon of energy $2E$

Electron $\longrightarrow$ positron $=$ electron with $E < 0$ going backward in time

Creation

Positron (electron arriving from future time) absorbs photon

Positron $\longrightarrow$ electron with $E > 0$ going forward in time

The Stueckelberg-Horwitz-Piron (SHP) Framework

Stueckelberg worldline theory of particles and antiparticles (1941)

Analytical mechanics of spacetime events

Equations of motion for event $x^\mu(\tau)$

$$\ddot{x}^\mu = F(x, \dot{x}, \tau)$$

Worldline $=$ τ-evolution of $x^\mu(\tau)$

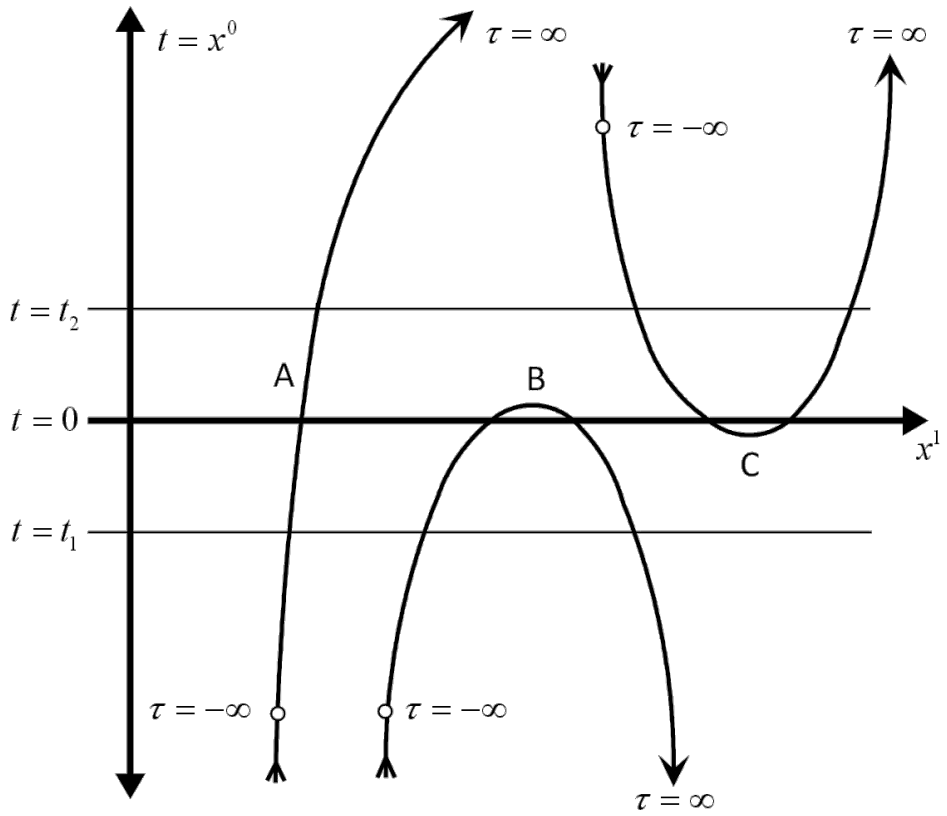
Coordinate $x^0 = ct$ may advance or retreat

External evolution parameter τ

- Monotonic replacement for x^0
- Not proper time of motion
  - Independent of $\dot{x}^\mu$
  - $\dot{x}^2 = \dot{x}^\mu \dot{x}_\mu$ changes sign on curves B and C
  - $\sqrt{-\dot{x}^2}$ not well-defined

Distinguishes two aspects of time

- $x^0 \sim$ coordinate time of an event
- $\tau \sim$ chronological ordering of a sequence of events



Worldlines

Type A. Usual type, with one solution to $t(\tau) = x^0$ for each x^0

Type B. Annihilation type, with two solutions to $t(\tau) = x^0$ for $x^0 \ll 0$ and no solutions for $x^0 \gg 0$

Type C. Creation type, with two solutions to $t(\tau) = x^0$ for $x^0 \gg 0$ and no solutions for $x^0 \ll 0$

The Stueckelberg-Horwitz-Piron (SHP) Framework

Horwitz-Piron Covariant Canonical Mechanics (1973)

Evolving 8D unconstrained phase space

$$x^\mu(\tau),\ \dot{x}^\mu(\tau) \qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau} \qquad \lambda, \mu, \nu, \ldots = 0, 1, 2, 3$$

Scalar Lagrangian $\longrightarrow$ manifestly covariant Euler-Lagrange equations

$$L = \frac{1}{2} M g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu - V(x) \qquad \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0$$

Canonical momentum

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = M g_{\mu\nu} \dot{x}^\nu \qquad \longrightarrow \qquad \dot{x}^\mu = \frac{1}{M} g^{\mu\nu} p_\nu$$

Scalar Hamiltonian $\longrightarrow$ manifestly covariant Hamilton equations

$$K = \dot{x}^\mu p_\mu - L = \frac{1}{2M} g^{\mu\nu} p_\mu p_\nu + V(x) \qquad \dot{x}^\mu = \frac{\partial K}{\partial p_\mu} \qquad \dot{p}_\mu = -\frac{\partial K}{\partial x^\mu}$$

Stueckelberg-Schrodinger equation

$$i\hbar \partial_\tau \psi(x, \tau) = K \psi(x, \tau)$$

The Stueckelberg-Horwitz-Piron (SHP) Framework

Free Particle in Curved Background Spacetime

Lagrangian with background metric $\partial_\tau g_{\mu\nu}(x) = 0$

$$L = \frac{1}{2} M g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu \qquad\qquad \lambda, \mu, \nu = 0, 1, 2, 3$$

Euler-Lagrange $\longrightarrow$ geodesic equations

$$0 = \frac{D\dot{x}^\lambda}{D\tau} = \ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \qquad\qquad \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\nu\mu} \right)$$

Hamiltonian $K =$ total mass — unconstrained but conserved on geodesic

$$K = \dot{x}^\mu p_\mu - L = \frac{1}{2M} g^{\mu\nu} p_\mu p_\nu \qquad\qquad \frac{dK}{d\tau} = M g_{\mu\nu} \dot{x}^\mu \frac{D\dot{x}^\nu}{D\tau} = 0$$

Poisson bracket

$$\frac{dK}{d\tau} = \{K, K\} + \frac{\partial K}{\partial \tau} = \frac{1}{2M} p_\mu p_\nu \, \partial_\tau g^{\mu\nu}(x) = 0$$

4+1 SHP Gravitation

Dynamical Local Metric $\gamma_{\mu\nu}(x,\tau)$

Spacetime tells matter how to move; matter tells spacetime how to curve.

— J. A. Wheeler

Matter

Particle = worldline traced out by event $x^\mu(\tau)$

Evolving events $x^\mu(\tau) \longrightarrow$ evolving mass density $\rho(x,\tau)$

Mass density $\rho(x,\tau) \longrightarrow$ energy-momentum tensor $T^{\mu\nu}(x,\tau)$

4D block universe $\mathcal{M}(\tau)$ **occurs irreversibly** at τ

Scalar Hamiltonian K generates evolution $\mathcal{M}(\tau) \longrightarrow \mathcal{M}(\tau + d\tau)$

Matter evolving in $\tau \longrightarrow$ curvature evolving in τ

Require generalized Einstein equations to find 4D metric $\gamma_{\mu\nu}(x,\tau)$

Flat spacetime $\gamma_{\mu\nu}(x,\tau) \longrightarrow \eta_{\mu\nu} = \text{diag}\,(-1,1,1,1)$

4+1 SHP Gravitation

Clues from SHP Electrodynamics

Classical action invariant under U(1) gauge transformations generated by $\Lambda(x,\tau)$

$$S_{\text{free}} \longrightarrow S_{\text{SHP}} = \int d\tau \; \frac{1}{2} M \dot{x}^{\mu} \dot{x}_{\mu} + \frac{e}{c} \dot{x}^{\mu} a_{\mu}\left(x,\tau\right) + \frac{e}{c} \dot{x}^{5} a_{5}\left(x,\tau\right)$$

$$= \int d\tau \; \frac{1}{2} M \dot{x}^{\mu} \dot{x}_{\mu} + \frac{e}{c} \dot{x}^{\alpha} a_{\alpha}\left(x,\tau\right)$$

$\alpha,\beta,\gamma = 0,1,2,3,5$ $\qquad$ $x^5 = c_5\tau$ in analogy to $x^0 = ct$ $\qquad$ $\dot{x}^5 =$ constant

S_{SHP} invariances

5D gauge invariance $\qquad a_{\alpha}\left(x,\tau\right) \longrightarrow a_{\alpha}\left(x,\tau\right) + \partial_{\alpha}\Lambda(x,\tau)$

4D Lorentz invariance $\qquad \dot{x}^{\mu} a_{\mu}$ and a_5 are O(3,1) scalars

S_{SHP} is standard 5D action with symmetry broken to O(3,1) in matter term

$$S_{5D} = \int d\tau \; \frac{1}{2} M \dot{x}^{\alpha} \dot{x}_{\alpha} + \frac{e}{c} \dot{x}^{\alpha} a_{\alpha} \xrightarrow[\dot{x}^5 \equiv c_5]{} S_{\text{SHP}} = \int d\tau \; \frac{1}{2} M \dot{x}^{\mu} \dot{x}_{\mu} + \frac{e}{c} \dot{x}^{\alpha} a_{\alpha}$$

Modified Lorentz force

$$M\ddot{x}^{\mu}\left(\tau\right) = \frac{e}{c} f^{\mu}{}_{\alpha}(x,\tau)\dot{x}^{\alpha}\left(\tau\right) \qquad \frac{d}{d\tau}(-\frac{1}{2}M\dot{x}^2) = \frac{ec_5}{c} f_{5\mu}\dot{x}^{\mu} \qquad f_{\alpha\beta} = \partial_{\alpha}a_{\beta} - \partial_{\beta}a_{\alpha}$$

4+1 SHP Gravitation

Free Particle in Curved Spacetime

Free particle action on 5D pseudo-spacetime $\mathcal{M}_5$ with coordinates $X^\alpha = (x^\mu, c_5\tau)$

$$S_{5D} = \int d\tau \, \frac{1}{2} M \dot{X}^\alpha \dot{X}_\alpha = \int d\tau \, \frac{1}{2} M g_{\alpha\beta}(x,\tau) \dot{X}^\alpha \dot{X}^\beta$$

Represents 5D interval $dX = X_1 - X_2 = (x_1, c_5\tau_1) - (x_2, c_5\tau_2)$

between events $x_1 \in \mathcal{M}(\tau_1)$ and $x_2 \in \mathcal{M}(\tau_2)$

5D equations of motion

$$\ddot{X}^\gamma + \Gamma^\gamma_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta = 0 \qquad \Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\beta\alpha} \right)$$

5D symmetry O(3,2) or O(4,1)

SHP equations of motion: break 5D symmetry to O(3,1)

Require $\dot{x}^5 = c_5$ non-dynamical (constant)

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0$$

$$\ddot{x}^5 \equiv 0$$

4+1 SHP Gravitation

SHP Einstein equations on $\mathcal{M}_5$

Non-interacting dust: mass density $= \rho(x,\tau)$ and pressure $p = 0$

5-component event current $= j^{\alpha}(x,\tau) = \dot{x}^{\alpha}(\tau)\rho(x,\tau)$

4-vector $\quad j^{\mu} = \dot{x}^{\mu}(\tau)\rho(x,\tau) \quad$ scalar $\quad j^{5}(x,\tau) = c_5\rho(x,\tau)$

Conserved current and mass-energy-momentum tensor

$$\nabla_{\alpha} j^{\alpha} = \partial_{\alpha} j^{\alpha} + j^{\gamma}\Gamma^{\alpha}_{\gamma\alpha} = \nabla_{\mu} j^{\mu} + \partial_{\tau}\rho = 0$$

$$T^{\alpha\beta} = \rho\dot{x}^{\alpha}\dot{x}^{\beta} \longrightarrow \left\{ \begin{array}{l} T^{\mu\nu} = \rho\dot{x}^{\mu}\dot{x}^{\nu} \\ T^{5\beta} = c_5 j^{\beta} \end{array} \right\} \longrightarrow \nabla_{\mu} T^{\alpha\mu} + \partial_5 T^{\alpha 5} = 0$$

SHP field equations

5D Einstein tensor satisfies 5D Bianchi identity $\nabla_{\alpha}\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right) = 0$

4+1 foliation breaks 5D symmetry $\longrightarrow$ O(3,1)

τ-evolution vector n_{α} normal to 4D spacetime hypersurface $\mathcal{M} \subset \mathcal{M}_5$

Projector $\quad P^{\alpha\beta} = g^{\alpha\beta} - \sigma n^{\alpha} n^{\beta} \quad$ onto $\mathcal{M} \qquad$ where $\quad n^2 = \sigma$

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}P^{\alpha\beta}R \longrightarrow \nabla_{\mu} G^{\alpha\mu} + \partial_5 G^{\alpha 5} = 0 \longrightarrow G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}$$

SHP field equations and conversation laws O(3,1) covariant

4+1 SHP Gravitation

Quick Overview of Arnowitt Deser Misner (ADM) Formalism

3+1 decomposition of 4D spacetime

Time evolution vector n_μ normal to 3D spacelike hypersurface Σ

Projection operator $P_{\mu\nu}$: 4D spacetime $\mathcal{M} \longrightarrow$ 3D space Σ

Induced 3D space metric on Σ: $\gamma_{ij} = P_{ij}$

Decomposition of geometrical objects and field equations

Projected covariant derivative D_μ and projected curvature $\overline{R}_{\mu\nu\lambda\rho}$ on Σ

Extrinsic curvature $K_{\mu\nu}$ = projected gradient of time vector n_μ

Decompose 4D Riemann tensor $R_{\mu\nu\lambda\rho} \longrightarrow$ combinations of $\overline{R}_{\mu\nu\lambda\rho}$ and $K_{\mu\nu}$

Decompose $T_{\mu\nu} \longrightarrow T_{ij}$, momentum p_i, energy density E

Initial value problem for space metric in canonical formulation of GR

Lie derivatives $\mathcal{L}_{n_\mu}$ along $n_\mu \longrightarrow$ PDEs for γ_{ij} and K_{ij} first order in ∂_t

Hamiltonian from canonical conjugates γ_{ij} and $\pi_{ij} = \sqrt{\gamma}\left(K_{ij} - \gamma_{ij}K\right)$

4+1 decomposition of 5D pseudo-spacetime $\mathcal{M}_5$

τ evolution vector n_α normal to 4D spacetime hypersurface $\mathcal{M}$

Projection operator $P_{\alpha\beta}$: 5D pseudo-spacetime $\mathcal{M}_5 \longrightarrow$ 4D spacetime $\mathcal{M}$

Tangent space $\mathcal{T}(\mathcal{M}_5)$ spanned by $\{\mathbf{g}_\mu\}$ and $\mathbf{g}_5 = N^\mu \mathbf{g}_\mu + Nn$

$\{\mathbf{g}_\mu\}$ spans $\mathcal{T}(\mathcal{M})$

Arbitrary shift N^μ and lapse $N \longrightarrow$ Lagrange multipliers

Metric decomposition

$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix} \qquad g^{\alpha\beta} = \begin{bmatrix} \gamma^{\mu\nu} + \sigma \frac{1}{N^2} N^\mu N^\nu & -\sigma \frac{1}{N^2} N^\mu \\ -\sigma \frac{1}{N^2} N^\mu & \sigma \frac{1}{N^2} \end{bmatrix}$$

Decomposition of mass-energy-momentum tensor

$$T_{\alpha\beta} = T_{\alpha'\beta'} \left(P_\alpha^{\alpha'} + \sigma n^{\alpha\alpha\prime} n_\alpha \right) \left(P_\beta^{\beta'} + \sigma n^{\beta'} n_\beta \right) = S_{\alpha\beta} - 2\sigma n_\alpha p_\beta + n_\alpha n_\beta \kappa$$

$$S_{\alpha\beta} \triangleq P_\alpha^{\alpha'} P_\beta^{\beta'} T_{\alpha'\beta'} \qquad p_\beta \triangleq -n^\alpha P_\beta^{\beta'} T_{\alpha\beta'} \qquad \kappa \triangleq n^\beta n^\alpha T_{\alpha\beta}$$

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Initial Value Problem

Generalize ADM prescription to project geometrical objects and field equations

Projected covariant derivative D_μ, intrinsic Ricci tensor $\overline{R}_{\mu\nu}$, extrinsic curvature $K_{\mu\nu}$

First order coupled evolution equations on 4D objects

$$\frac{1}{c_5}\partial_\tau \gamma_{\mu\nu} = \mathcal{L}_{\mathbf{N}}\, \gamma_{\mu\nu} - 2NK_{\mu\nu}$$

$$\frac{1}{c_5}\partial_\tau K_{\mu\nu} = \mathcal{L}_{\mathbf{N}} K_{\mu\nu} - \sigma D_\mu D_\nu N + N\left[-\sigma \overline{R}_{\mu\nu} + KK_{\mu\nu} - 2K^\lambda_\mu K_{\nu\lambda} + \sigma\frac{8\pi G}{c^4}\left(S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}S\right)\right]$$

$$D_\mu \triangleq P^\alpha_\mu P^\gamma_\alpha \nabla_\gamma \qquad \left[D_\nu, D_\mu\right] X_\lambda \triangleq X_\rho\, \overline{R}^\rho_{\lambda\mu\nu}$$

$$K_{\mu\nu} \triangleq -P^\alpha_\mu P^\beta_\nu \nabla_\beta n_\alpha = -\nabla_\mu n_\nu - n_\mu \frac{1}{N} D_\nu N$$

Non-evolving constraints

$$\overline{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) = -\frac{8\pi G}{c^4}\left(S + \sigma\kappa\right) \qquad D_\mu K^\mu_\nu - D_\nu K = \frac{8\pi G}{c^4} p_\nu$$

Lapse N and shift N_μ are non-dynamical

Propagate with τ

No PDEs describing τ-evolution

4+1 SHP Gravitation

Example — Modified Schwarzschild Metric

Evolving τ-dependent metric

$$\gamma_{\mu\nu} = e^{-\lambda\beta(t-\tau)}\,\mathrm{diag}\left(-U(r), \frac{1}{U(r)}, r^2, r^2\sin^2\theta\right) \qquad U(r) = 1 - \frac{2MG}{c^2 r}$$

$\beta(t-\tau)$ is some even function

Test event at $x = (ct, \boldsymbol{x})$ feels metric centered at $\tau = t$

Test event evolving as $x(\tau) = (c\tau, \boldsymbol{x}_0)$ sees standard Schwarzschild metric

Geodesic equations may divert event from trajectory $t(\tau) = \tau$

Perturbation theory to order $o(\lambda)$

$$e^{-\lambda\beta} \simeq 1 - \lambda\beta \;\longrightarrow\; \gamma_{\mu\nu} = \gamma^{(0)}_{\mu\nu} - \lambda\gamma^{(1)}_{\mu\nu} \qquad \gamma^{(1)}_{\mu\nu} = \beta\gamma^{(0)}_{\mu\nu}$$

$$\Gamma^{\rho}_{\mu\nu} = \Gamma^{(0)\rho}_{\mu\nu} + \lambda\Gamma^{(1)\rho}_{\mu\nu}$$

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + \lambda R^{(1)}_{\mu\nu} = \lambda R^{(1)}_{\mu\nu}$$

$$S_{\mu\nu} = S^{(0)}_{\mu\nu} + \lambda S^{(1)}_{\mu\nu} = \lambda S^{(1)}_{\mu\nu}$$

$$\gamma^{(0)}_{\mu\nu} = \mathrm{diag}\left(-U(r), \frac{1}{U(r)}, r^2, r^2\sin^2\theta\right)$$
$$R^{(0)}_{\mu\nu} = 0$$
$$S^{(0)}_{\mu\nu} = 0$$

4+1 SHP Gravitation

Example — Initial Value Problem for Perturbed Metric

Initial value problem taking $N = 1$ and $N_\mu = 0$

$$\frac{1}{c_5}\partial_\tau\left(\gamma^{(0)}_{\mu\nu} + \lambda\gamma^{(1)}_{\mu\nu}\right) = -2\left(K^{(0)}_{\mu\nu} + \lambda K^{(1)}_{\mu\nu}\right)$$

$$\frac{1}{c_5}\partial_\tau\left(K^{(0)}_{\mu\nu} + \lambda K^{(1)}_{\mu\nu}\right) = -\sigma\overline{R}_{\mu\nu} + KK_{\mu\nu} - 2K^\lambda_\mu K_{\nu\lambda} + \sigma\frac{8\pi G}{c^4}\left(S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}S\right)$$

Extrinsic curvature

$$K_{\mu\nu} = K^{(0)}_{\mu\nu} + \lambda K^{(1)}_{\mu\nu} \propto \partial_\tau\gamma_{\mu\nu} = \partial_\tau\left(\gamma^{(0)}_{\mu\nu} + \lambda\gamma^{(1)}_{\mu\nu}\right) = \lambda\partial_\tau\gamma^{(1)}_{\mu\nu} \longrightarrow K_{\mu\nu} = \lambda K^{(1)}_{\mu\nu}$$

Neglect $KK_{\mu\nu} - 2K^\lambda_\mu K_{\nu\lambda} = o\left(\lambda^2\right)$

Evolution of perturbed system

Using $\overline{R}^{(0)} = S^{(0)}_{\mu\nu} = 0$ for Schwarzschild

$$\frac{1}{c_5}\partial_\tau\gamma^{(1)}_{\mu\nu} = -2K^{(1)}_{\mu\nu} \qquad \frac{1}{c_5}\partial_\tau K^{(1)}_{\mu\nu} = -\sigma\overline{R}^{(1)}_{\mu\nu} + \sigma\frac{8\pi G}{c^4}\left(S^{(1)}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}S^{(1)}\right)$$

Using initial $\gamma^{(1)}_{\mu\nu} \longrightarrow \overline{R}^{(1)}_{\mu\nu}$ calculated from $\gamma^{(1)}_{\mu\nu}$, and some choice for $K^{(1)}_{\mu\nu}$

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Example — Perturbed Curvature

Ricci tensor

$$\overline{R}^{(1)}_{\mu\nu} = \ddot{\beta}\ \mathrm{diag}\left(\frac{3}{2}, \frac{1}{2U^2}, \frac{r^2}{2U}, \frac{r^2\sin^2\theta}{2U}\right)$$

$$+\ \dot{\beta}\begin{bmatrix} 0 & \frac{5}{4}\frac{\partial_r U}{U}+\frac{1}{r} & \frac{\partial_r U}{U}-\frac{1}{r}U^2+\frac{1}{r} & \frac{\partial_r U}{U}+\frac{1}{r} \\ \frac{5}{4}\frac{\partial_r U}{U}+\frac{1}{r} & 0 & 0 & 0 \\ \frac{\partial_r U}{U}-\frac{1}{r}U^2+\frac{1}{r} & 0 & 0 & 0 \\ \frac{\partial_r U}{U}+\frac{1}{r} & 0 & 0 & 0 \end{bmatrix}$$

where $\dot{\beta} = \partial_5\beta\,(t-\tau)$

Ricci scalar

$$\overline{R}^{(1)} = \gamma^{\mu\nu}_{(0)}\overline{R}^{(1)}_{\mu\nu} = \left(\ddot{\beta}/2U\right)(-3+1+1+1) = 0$$

Hamiltonian constraint

Taking $\kappa = 0$ and using $\overline{R} = \overline{R}^{(0)} + \lambda\overline{R}^{(1)} = 0$ and $K^2 - K^{\mu\nu}K_{\mu\nu} = o\left(\lambda^2\right)$

$$\overline{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) = -\frac{8\pi G}{c^4}\left(S + \sigma\kappa\right) \longrightarrow S = 0$$

Evolution equations

$$\frac{1}{c_5}\partial_\tau \gamma^{(1)}_{\mu\nu} = -2K^{(1)}_{\mu\nu} \qquad \frac{1}{c_5}\partial_\tau K^{(1)}_{\mu\nu} = -\sigma \overline{R}^{(1)}_{\mu\nu} + \sigma \frac{8\pi G}{c^4} S^{(1)}_{\mu\nu}$$

Initial form of extrinsic curvature

$\mu \neq \nu$ $\quad \gamma^{(1)}_{\mu\nu} = 0 \longrightarrow$ choose $K^{(1)}_{\mu\nu} = 0$

Set $\frac{8\pi G}{c^4} S^{(1)}_{\mu\nu} = \overline{R}^{(1)}_{\mu\nu} \longrightarrow \partial_\tau K^{(1)}_{\mu\nu} = 0$

Off-diagonal components of $\gamma^{(1)}_{\mu\nu}$ and $K^{(1)}_{\mu\nu}$ do not evolve

$\mu = \nu$ $\quad$ Choose some initial $K^{(1)}_{\mu\nu}$ and $S^{(1)}_{\mu\nu}$

Subject to $S^{(1)} = \gamma^{\mu\nu}_{(0)} S^{(1)}_{\mu\nu} = 0$

Evolution equations are diagonal

Leads to initial value problem for $\beta(t - \tau)$

4+1 SHP Gravitation

Example — Toy Solution

Must have $\tilde{S}^{(1)} = 0$ but free to choose $\tilde{S}^{(1)}_{\mu\nu} = 0 \longrightarrow$ no matter source

$$\partial_5 \gamma^{(1)}_{\mu\nu} = -2K^{(1)}_{\mu\nu} \qquad \partial_5 K^{(1)}_{\mu\nu} = -\sigma \overline{R}^{(1)}_{\mu\nu} \qquad \mu = \nu$$

Integrate evolution equations

$$\begin{aligned} K^{(1)}_{\mu\nu} &= -\sigma \int_{-\infty}^{x^5} \overline{R}^{(1)}_{\mu\nu}(x,s)\, ds = -\sigma \int_{-\infty}^{x^5} \ddot{\beta}\ \mathrm{diag}\left(\frac{3}{2}, \frac{1}{2U^2}, \frac{r^2}{2U}, \frac{r^2\sin^2\theta}{2U}\right) ds \\ &= \sigma\dot{\beta}\ \mathrm{diag}\left(\frac{3}{2}, \frac{1}{2U^2}, \frac{r^2}{2U}, \frac{r^2\sin^2\theta}{2U}\right) \\ \gamma^{(1)}_{\mu\nu} &= -2\int_{-\infty}^{x^5} K^{(1)}_{\mu\nu}(x,s)\, ds = -2\sigma \int_{-\infty}^{x^5} \dot{\beta}\ \mathrm{diag}\left(\frac{3}{2}, \frac{1}{2U^2}, \frac{r^2}{2U}, \frac{r^2\sin^2\theta}{2U}\right) ds \\ &= \frac{\sigma\beta}{U}\mathrm{diag}\left(3U, \frac{1}{U}, r^2, r^2\sin^2\theta\right) \end{aligned}$$

Perturbed metric leads to the interval

$$dx^2 = e^{-3R}\left(-c^2 U\, dt^2\right) + e^{-R}\left(\frac{1}{U}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)$$

where $R(t,r,\tau) = \lambda\dfrac{(t-\tau)^2}{U(r)}$ and $U = 1 - \dfrac{2MG}{c^2 r}$

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Example — Geodesic Equations of Motion

Euler-Lagrange equations using rotational symmetry to fix $\theta = \pi/2$

$$\ddot{t} = -\dot{t}\dot{r}\frac{\partial_r U}{U} - \frac{1}{2}\lambda\dot{\beta}\left(c\dot{t}^2 - \dot{t} + \frac{1}{U^2}\frac{\dot{r}^2}{c}\right)$$

$$\ddot{r} = \frac{1}{2}\frac{\partial_r U}{U}\dot{r}^2 - \frac{1}{2}c^2 U\left(\partial_r U\right)\dot{t}^2 - \lambda\dot{\beta}\dot{r}\left(c\dot{t} - 1\right)$$

$$\ddot{\phi} = -\frac{2\dot{r}}{r}\dot{\phi} - \lambda\dot{\beta}\dot{\phi}\left(c\dot{t} - 1\right)$$

$$\dot{} = \partial_5 = \frac{1}{c_5}\partial_\tau$$

Perturbation breaks angular momentum conservation

$$L = mr^2\dot{\phi} \longrightarrow \frac{d}{d\tau}L = mr\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) = -\lambda\dot{\beta}\left(c\dot{t} - 1\right)L$$

Test event starting at $t = \tau$

Unperturbed time equation $\ddot{t} = -\dot{t}\dot{r}\frac{\partial_r U}{U}$ $\longrightarrow$ time acceleration

Event evolves away from initial trajectory

Perturbation $\lambda\dot{\beta}(t - \tau)$ to Schwarzschild comes into effect

4+1 SHP Gravitation

Weak Gravitation in the SHP Formalism

$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ $\quad$ $h_{\alpha\beta}$ = small perturbation of 5D flat metric

5D wave equation $\quad -\frac{1}{2}\partial^{\gamma}\partial_{\gamma}h_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}\widehat{\eta}_{\alpha\beta}S\right)$

where $\widehat{\eta}_{\alpha\beta} = \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\,\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1,0)$

Metric perturbation $\longrightarrow$ foliation

Lapse $N = 1 + \frac{1}{2}\sigma h_{55}$ $\quad$ Shift $N_{\mu\nu} = h_{5\mu}$ $\quad$ Normal $n_{\alpha} = \sigma\left[1 + \frac{1}{2}\sigma h_{55}\right]\delta^{5}_{\alpha}$

Extrinsic curvature recovers 5-component of connection

$$K_{\mu\nu} = -\nabla_{\mu}n_{\nu} - n_{\mu}\frac{1}{N}D_{\nu}N = \frac{1}{2}\left(\partial_{\nu}h_{5\mu} + \partial_{\mu}h_{5\nu} - \partial_{5}h_{\mu\nu}\right) = \Gamma_{5\mu\nu}$$

Initial value problem

$$\frac{1}{c_5}\partial_{\tau}\gamma_{\mu\nu} = -2K_{\mu\nu} + \partial_{\mu}h_{5\nu} + \partial_{\nu}h_{5\mu} \qquad \partial_{\mu}K^{\mu}_{\nu} - \partial_{\nu}K = \frac{8\pi G}{c^4}p_{\nu}$$

$$\frac{1}{c_5}\partial_{\tau}K_{\mu\nu} = \frac{1}{2}\partial_{\mu}\partial_{\nu}h_{55} - \sigma\left[\overline{R}_{\mu\nu} - \frac{8\pi G}{c^4}\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S\right)\right] \qquad \overline{R} = -\frac{8\pi G}{c^4}\left(S + \sigma\kappa\right)$$

4+1 SHP Gravitation

Equilibrium in Weak Gravitation

Initial value problem under equilibrium condition $\partial_\tau \longrightarrow 0$

$$(1)\quad 0 = -2K_{\mu\nu} + \partial_\mu h_{5\nu} + \partial_\nu h_{5\mu} \qquad (3)\quad \partial_\mu K^\mu_\nu - \partial_\nu K = \frac{8\pi G}{c^4} p_\nu$$

$$(2)\quad 0 = \frac{1}{2}\partial_\mu\partial_\nu h_{55} - \sigma\left[\overline{R}_{\mu\nu} - \frac{8\pi G}{c^4}\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S\right)\right] \qquad (4)\quad \overline{R} = -\frac{8\pi G}{c^4}(S + \sigma\kappa)$$

Equation (1) satisfied automatically by $K_{\mu\nu} = \Gamma_{5\mu\nu}$ with $\partial_\tau h_{\mu\nu} = 0$

Constraints (3) and (4) satisfied automatically by evolution and wave equations

Consequences of non-zero mass density κ

Wave equation for $\kappa = n^\beta n^\alpha T_{\alpha\beta} \simeq T_{55} \longrightarrow -\frac{1}{2}\partial^\alpha\partial_\alpha h_{55} = -\frac{1}{2}\partial^\mu\partial_\mu h_{55} = \frac{8\pi G}{c^4}\kappa$

$\kappa \longrightarrow h_{55}$ using 4D Green's function

Denote $\frac{1}{2}\partial_\mu\partial_\nu h_{55} = \frac{8\pi G}{c^4}\left(\widehat{S}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\widehat{S}\right) \longrightarrow \widehat{S} = \kappa$

Equation (2) becomes $\overline{R}_{\mu\nu} = \frac{8\pi G}{c^4}\left[\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S\right) + \sigma\left(\widehat{S}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\widehat{S}\right)\right]$

Standard Einstein equations with $\kappa \longrightarrow$ extra 'dark matter' or 'dark energy' terms

Kaluza-Klein Theory

Proposed Unification of Electromagnetism and Gravitation (1919, 1926)

Extend Einstein general relativity (GR) to five dimensions

$$dX^2 = g_{\alpha\beta} dX^\alpha dX^\beta \qquad x^\alpha = x^0, x^1, x^2, x^3, x^5$$

Index convention $\alpha, \beta = 0, 1, 2, 3, 5$, and $\mu, \nu = 0, 1, 2, 3$

5D Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 0$$

Decompose metric

$$dX^2 = \gamma_{\mu\nu} dx^\mu dx^\nu + \phi^2 \left(A_\mu dx^\mu + dx^5 \right)^2$$

Decompose 5D Einstein equations

$$S = \int d^4x dx^5 \sqrt{-g} \left(\overline{R} + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right)$$

where $\overline{R} = g^{\mu\nu} R_{\mu\nu}$ $\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Requirements for identification of $F_{\mu\nu}$ with Maxwell field

$\phi =$ constant $\qquad \partial_5 \gamma_{\mu\nu} = \partial_5 A_\mu = 0 \qquad x^5$ compactified

Kaluza-Klein Theory

Kaluza-Klein Decomposition in SHP Theory

Rewrite interval in more symmetric form

$$dX^2 = g_{\alpha\beta} dX^\alpha dX^\beta = \gamma_{\mu\nu} dx^\mu dx^\nu + \left(a_\mu dx^\mu + a_5 dx^5\right)^2$$

$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} + a_\mu a_\nu & a_\mu a_5 \\ a_\mu a_5 & a_5^2 \end{bmatrix} \qquad g^{\alpha\beta} = \begin{bmatrix} \gamma^{\mu\nu} & -\frac{1}{a_5} a^\mu \\ -\frac{1}{a_5} a^\nu & \frac{1}{a_5^2}\left(1 + a_\mu a^\mu\right) \end{bmatrix}$$

Comparing with 4+1 SHP metric

$$g^{\alpha\beta} = \begin{bmatrix} \gamma^{\mu\nu} + \sigma \frac{1}{N^2} N^\mu N^\nu & -\sigma \frac{1}{N^2} N^\mu \\ -\sigma \frac{1}{N^2} N^\mu & \sigma \frac{1}{N^2} \end{bmatrix} \qquad g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix}$$

Writing $\frac{1}{a_5^2} = \sigma N^2$ and $a_5 a_\mu = -N^\mu \implies \left(g^{\text{Kaluza-Klein}}\right)^{\alpha\beta} = \left(g^{\text{SHP}}\right)_{\alpha\beta}$

But a_μ and a_5 are dynamical fields while N_μ and N are non-dynamical

Kaluza-Klein Theory

Geodesic Equations

5D interval $\longrightarrow$ action $S = \int d\tau\ L\left(X, \dot{X}, \tau\right) = \int d\tau\ \frac{1}{2} M g_{\alpha\beta} \dot{X}^{\alpha} \dot{X}^{\beta}$

Starting assumptions

$g_{\alpha\beta} = g_{\alpha\beta}(x, \tau)$ for all α, β

No compact dimensions

Euler-Lagrange equations $\longrightarrow$ $M\ddot{X}^{\gamma} + \Gamma^{\gamma}_{\alpha\beta} \dot{X}^{\alpha} \dot{X}^{\beta} = 0$

Spacetime components expand as

$$M\ddot{x}^{\mu} + \Gamma^{\mu}_{\lambda\rho} \dot{x}^{\lambda} \dot{x}^{\rho} + 2\Gamma^{\mu}_{5\lambda} \dot{x}^{5} \dot{x}^{\lambda} + \Gamma^{\mu}_{55} \dot{x}^{2}_{5} = 0$$

$\gamma = 5$ components

Standard SHP GR $\longrightarrow$ $x^5 = c_5 \tau$ $\longrightarrow$ $\ddot{x}^5 \equiv 0$

Here we follow Kaluza and Klein

$$\left[\frac{d}{d\tau} \frac{\partial}{\partial \dot{x}^5} - \frac{\partial}{\partial x^5}\right] L = \frac{d}{d\tau} \left(a_{\mu} a_5 \dot{x}^{\mu} + a_5^2 \dot{x}^5\right) - \frac{1}{2} \left(\partial_5 g_{\alpha\beta}\right) \dot{x}^{\alpha} \dot{x}^{\beta} = 0$$

Kaluza-Klein Theory

Evaluating the Connection

Metric

$$g_{\alpha\beta} = \overline{\gamma}_{\alpha\beta} + a_\alpha a_\beta \qquad \text{where} \qquad \overline{\gamma}_{\alpha\beta} = \delta^\lambda_\alpha \delta^\rho_\beta \gamma_{\lambda\rho} \longrightarrow g_{5\beta} = a_5 a_\beta$$

Decompose connection into 4D metric and field terms

$$\Gamma^\gamma_{\alpha\beta} = \overline{\Gamma}^\gamma_{\alpha\beta} + \frac{1}{2} g^{\gamma\delta} \left[a_\delta \left(\partial_\beta a_\alpha + \partial_\alpha a_\beta \right) + a_\alpha f_{\beta\delta} + a_\beta f_{\alpha\delta} \right]$$

where $f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha$

and $\overline{\Gamma}_{\gamma\alpha\beta} = \frac{1}{2} \left[\partial_\beta \overline{\gamma}_{\alpha\gamma} + \partial_\alpha \overline{\gamma}_{\beta\gamma} - \partial_\gamma \overline{\gamma}_{\alpha\beta} \right]$

Decompose $\overline{\Gamma}^\mu_{\alpha\beta}$ into spacetime and 5 components

$$\overline{\Gamma}^\mu_{\lambda\rho} = \Gamma^{(4)\mu}_{\lambda\rho} - \frac{1}{2} a^\mu a^5 \partial_5 \gamma_{\lambda\rho} \qquad \overline{\Gamma}^\mu_{\nu 5} = \frac{1}{2} g^{\mu\lambda} \partial_5 \gamma_{\nu\lambda} \qquad \overline{\Gamma}^\mu_{55} = 0$$

with the standard 4D Christoffel symbol

$$\Gamma^{(4)\mu}_{\lambda\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\lambda g_{\sigma\rho} + \partial_\rho g_{\sigma\lambda} - \partial_\sigma g_{\rho\lambda} \right)$$

Kaluza-Klein Theory

Kaluza-Klein Conditions on Metric

5 connection components in spacetime geodesic equation

$$\Gamma^{\mu}_{\lambda 5} = \frac{1}{2} g^{\mu\lambda} \partial_5 \gamma_{\nu\lambda} + \frac{1}{2} \left[a^{\mu} \left(\partial_5 a_{\lambda} + \partial_{\lambda} a_5 \right) + a_{\lambda} f_5{}^{\mu} + a_5 f_{\lambda}{}^{\mu} \right]$$

$$\Gamma^{\mu}_{55} = a^{\mu} \partial_5 a_5 + a_5 f_5{}^{\mu}$$

Contribution of 5-components to spacetime geodesic equations

$$\begin{aligned} 2c_5 \Gamma^{\mu}_{5\lambda} \dot{x}^{\lambda} + c_5^5 \Gamma^{\mu}_{55} \;=\; & \left(\gamma^{\mu\lambda} + a^{\mu} a^{\lambda} \right) \dot{x}^{\lambda} \partial_{\tau} \gamma_{\nu\lambda} \\ & + 2c_5 \left[a^{\mu} \partial_{\lambda} a_5 + \frac{1}{2} \left(a^{\mu} f_{5\lambda} + a_{\lambda} f_5{}^{\mu} + a_5 f_{\lambda}{}^{\mu} \right) \right] \dot{x}^{\lambda} \\ & + c_5^5 \left(a^{\mu} \partial_5 a_5 + a_5 f_5{}^{\mu} \right) \end{aligned}$$

Requirements for invariance under U(1) gauge transformation $a_{\alpha} \longrightarrow a_{\alpha} + \partial_{\alpha} \Lambda$

$a^{\mu} a^{\lambda}$ in first line $\longrightarrow$ $\partial_{\tau} \gamma_{\nu\lambda} = 0$

$a^{\mu} \partial_{\lambda} a_5$ in line 2 $\longrightarrow$ $\partial_{\lambda} a_5 = 0$

Equivalent to SHP electrodynamics in τ-equilibrium

$a^{\mu} f_{5\lambda}$ and $a^{\mu} \partial_5 a_5$ in lines 2,3 $\longrightarrow$ $\partial_{\tau} a_{\alpha} = 0$ $\longrightarrow$ $f^{\mu}_{5} = \partial_{\tau} a^{\mu} - \partial^{\mu} a_5 = 0$

Kaluza-Klein Theory

Equations of Motion

Kaluza-Klein conditions

$a_5 = \text{constant}$ $\qquad$ $\partial_\tau g_{\alpha\beta} = \partial_\tau \gamma_{\alpha\beta} + \partial_\tau \left(a_\alpha a_\beta\right) = 0$

$\partial_\tau \gamma_{\alpha\beta} = 0 \longrightarrow$ basis vector in 5-direction a Killing vector $\longrightarrow \dfrac{dP_5}{d\tau} = 0$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^5} = a_5 \frac{d}{d\tau}\left(a_\mu \dot{x}^\mu + a_5 \dot{x}^5\right) = 0 \longrightarrow a_\mu \dot{x}^\mu + a_5 \dot{x}^5 = \text{constant} = \frac{e}{c}$$

Field $a_\mu \longrightarrow$ Maxwell potential A_μ and field strength $F_{\mu\nu}$

Contributions of connection reduce to

$$\Gamma^\mu_{\lambda\rho}\dot{x}^\lambda \dot{x}^\rho = \Gamma^{(4)\mu}_{\lambda\rho}\dot{x}^\lambda \dot{x}^\rho + \frac{1}{2}\left(A_\lambda F_\rho{}^\mu + A_\rho F_\lambda{}^\mu\right)\dot{x}^\lambda \dot{x}^\rho$$

$$2c_5\Gamma^\mu_{5\lambda}\dot{x}^\lambda + c_5^5 \Gamma \dot{x}^\lambda = 2c_5 a_5 f_\lambda{}^\mu \dot{x}^\lambda$$

Equations of motion

$$0 = M\ddot{x}^\mu + \Gamma^{(4)\mu}_{\lambda\rho}\dot{x}^\lambda \dot{x}^\rho + \left(A_\lambda \dot{x}^\lambda + c_5 a_5\right) F_\rho{}^\mu \dot{x}^\rho = M\ddot{x}^\mu + \Gamma^{(4)\mu}_{\lambda\rho}\dot{x}^\lambda \dot{x}^\rho + \frac{e}{c}F_\rho{}^\mu \dot{x}^\rho$$

Standard 4D Lorentz force in curved spacetime

Kaluza-Klein Theory

Action for Metric and Fields

Connection under Kaluza-Klein conditions

$$\Gamma^{\mu}_{\lambda\rho} = \frac{1}{2}\gamma^{\mu\sigma}\left(\partial_\rho\gamma_{\sigma\lambda} + \partial_\lambda\gamma_{\sigma\rho} - \partial_\sigma\gamma_{\lambda\rho}\right) + \frac{1}{2}\left(A_\lambda F_\rho{}^{\mu} + A_\rho F_\lambda{}^{\mu}\right)$$

$$\Gamma^{\mu}_{\lambda 5} = a_5 F_\lambda{}^{\mu} \qquad\qquad \Gamma^{5}_{55} = 0$$

Diagonal components of Ricci tensor

$$R_{\mu\nu} = R^{(4)}_{\mu\nu} - \frac{1}{2}\left(A_\mu\nabla_\lambda F^\lambda_\nu + A_\nu\nabla_\lambda F^\lambda_\mu\right) + \frac{1}{4}\left(F^\lambda_\nu F_{\lambda\mu} + F^\lambda_\mu F_{\lambda\nu}\right) - \frac{1}{4}A_\mu A_\nu F_{\lambda\rho}F^{\lambda\rho}$$

$$R_{55} = \frac{1}{4}F_{\lambda\rho}F^{\lambda\rho}$$

5D Ricci tensor

$$R = R^{\alpha}_{\alpha} = R^{(4)} + \frac{1}{4}F_{\lambda\rho}F^{\lambda\rho} \qquad \text{where } R^{(4)} = \text{standard 4D Ricci scalar}$$

Einstein-Hilbert action

$$S = \int d^5X\sqrt{-g}R = \int d^5X\sqrt{-g}\left(R^{(4)} + \frac{1}{4}F_{\lambda\rho}F^{\lambda\rho}\right)$$

Thank You
For Your Patience

And The Suspension
Of Disbelief

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